

Characteristic length scale and energy of a vortex line in a dilute, superfluid Fermi gas

Ø. Elgarøy

Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

We calculate the characteristic length scale of a vortex line in a dilute, superfluid gas of fermionic atoms, and find that it is in general smaller, and has a weaker density dependence than the BCS coherence length. Taking this into account, we find the energy of a vortex line to be larger than has previously been estimated. As a consequence, the critical frequency for formation of vortices in trapped fermion gases can be significantly larger than earlier calculations have suggested.

The transition to a superfluid Bardeen-Cooper-Schrieffer (BCS) state at low temperatures is a generic feature of Fermi systems with an attractive effective interaction. The low-temperature superconductors and liquid ^3He are the most familiar examples, but the phenomenon can also be seen in more exotic systems like atomic nuclei, and superfluids are also thought to be important in neutron stars [1]. Shortly after Bose-Einstein condensation in a dilute gas of ^{87}Rb atoms was achieved [2], the possibility of observing the BCS transition to a superfluid state in a dilute gas of trapped fermionic atoms was proposed [3]. Trapping and cooling of fermionic alkali atoms have now been achieved, reaching temperatures as low as $\sim T_F/4$ for ^{40}K [4,5] and ^6Li [6–8], where T_F is the Fermi temperature. The typical transition temperature for weakly interacting fermions is $T_c \sim 10^{-3}T_F$, but one hopes to create systems with $T_c \sim 10^{-1}T_F$ in the experiments. If temperatures below T_c can be reached, a new and exciting laboratory for studying properties of fermion superfluids will be at our disposal. Finding experimental signatures of the BCS transition in these systems is therefore important, and considerable effort has been invested in this problem [9].

The study of quantized vortices in Bose-Einstein condensates has led to many interesting results [10], and recently the vortex state in trapped fermionic gases has also been studied [11,12] as a means for detecting a superfluid state in these systems. A fundamental quantity in these considerations is the critical rotational frequency for the creation of a vortex line with one quantum of circulation. The gases in the experiments are inhomogeneous because of the trapping potential, but if the vortex is produced near the center of the trap, one can estimate the critical frequency starting from the energy per unit length, \mathcal{E}_v , of a vortex in homogeneous matter [12,13]. Recently, Bruun and Viverit [12] calculated \mathcal{E}_v at zero temperature, assuming that the size of the vortex core, the region where the order parameter drops from its value in homogeneous matter to zero, is given by the BCS coherence length ξ_0 . However, several studies of vortices in low-temperature superconductors [14–16] have found that the formation of bound quasiparticle states localized near the center of

the vortex can cause the size of the vortex core to be much smaller than ξ_0 at low temperatures. The same effect has been found for vortices in superfluid neutron matter [17,18], and we will in this Letter argue that it is also present in dilute gases, and that this can lead to a significantly larger vortex energy than that found in [12].

We consider a dilute, homogeneous gas of fermionic atoms in two different hyperfine states, held at temperature $T = 0$. In this limit the Pauli principle dictates that the effective s -wave interaction between two atoms in the same hyperfine state vanish. The interaction between two atoms in different states can be approximated by the s -wave scattering length a . For a negative scattering length, the interaction is attractive and if the number of particles in the two states is the same, the $T = 0$ ground state of the gas is superfluid. Theoretically, the formation of a superfluid state is signaled by a non-zero value for the energy gap Δ_0 , which is proportional to the critical temperature T_c . In the dilute gas limit $k_F|a| \ll 1$, where k_F is the Fermi wavenumber, it is given by

$$\Delta_0 = \left(\frac{2}{e}\right)^{7/3} \epsilon_F e^{-\pi/2k_F|a|}, \quad (1)$$

where $\epsilon_F = \hbar^2 k_F^2 / 2m_a$, m_a being the mass of an atom, is the Fermi energy [19,20]. Corrections from the so-called induced interaction modify the numerical prefactor in this result, but the dependence on k_F and $|a|$ remains the same [19,20].

The energy per unit length of a vortex line with respect to the homogeneous ground state of the superfluid can be estimated by adding the kinetic energy associated with the velocity field around the vortex, and the loss of condensation energy in the vortex core [12]. In this approximation the vortex core is treated as a cylindrical column of normal matter with radius equal to the BCS coherence length $\xi_0 = \hbar v_F / \pi \Delta_0$, where $v_F = \hbar k_F / m_a$. The result is

$$\mathcal{E}_v = \frac{\pi \hbar^2 n_\sigma}{2m_a} \ln \left(D \frac{R_c}{\xi_0} \right), \quad (2)$$

where $n_\sigma = k_F^3 / 6\pi^2$ is the number density of one hyperfine state, and $D = 1.36$. The energy can also be

calculated from Ginzburg-Landau theory, and the result turns out to be of the same form, but with $D = 1.65$ [12], again assuming that ξ_0 is the characteristic length scale for the vortex. However, at low temperatures the microscopic properties of the system are important, and, as we will argue below, the existence of quasiparticle states localized near the center of the vortex has to be taken into account at very low temperatures.

The microscopic properties of vortices in fermion superfluids can be derived from the Bogoliubov-de Gennes (BdG) equations [21], which we write as

$$\mathcal{H}_{\text{BdG}} \Psi_i(\mathbf{r}) = E_i \Psi_i(\mathbf{r}), \quad (3)$$

where

$$\mathcal{H}_{\text{BdG}} = \begin{pmatrix} H_o(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_o^*(\mathbf{r}) \end{pmatrix}, \quad (4)$$

with $H_o = -\hbar^2 \nabla^2 / 2m_a - \epsilon_F$, and $\Psi_i = (u_i, v_i)^T$, where u_i and v_i are the coefficients in the Bogoliubov quasiparticle transformation, and i symbolizes the quantum numbers characterizing the quasiparticle states. The order parameter $\Delta(\mathbf{r})$ at temperature T is determined by the self-consistency condition

$$\Delta(\mathbf{r}) = \frac{4\pi\hbar^2|a|}{m_a} \sum_i u_i(\mathbf{r}) v_i^*(\mathbf{r}) \tanh\left(\frac{E_i}{2k_B T}\right), \quad (5)$$

where k_B is Boltzmann's constant. In cylindrical coordinates (ρ, φ, z) , a vortex with one quantum of circulation can be described by a complex order parameter $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{i\theta(\mathbf{r})}$. For a straight vortex line along the axis of the container, $|\Delta(\mathbf{r})| = \Delta(\rho)$, and $\theta = -\varphi$, where $\Delta(\rho)$ starts out at zero in the center of the vortex core at $\rho = 0$, and increases asymptotically to the value in homogeneous matter Δ_0 . The low-lying quasiparticle states can be obtained analytically, as first shown by Caroli *et al.* [22]. Kramer and Pesch [14] used a similar approach in a study of vortices in type II superconductors, where they showed that at low temperatures, a new length scale, different from ξ_0 , characterizes the size of the vortex core. Their calculation, translated to our notation, is as follows: We label the lowest eigenstates of the BdG Hamiltonian by their angular momentum μ , equal to half an odd integer, and the z component of the momentum $k_z = k_F \sin \theta$, where θ is the angle between \mathbf{k}_F and the xy plane. For $0 < \mu \ll k_F \xi_0$, where $k_F \xi_0$ is of order 10-100 in the weak coupling regime, and for small ρ , one finds that these eigenstates are of the form [14,21,22]

$$u_\mu(\mathbf{r}) = \left(\frac{k_F \Delta_0}{2L_z v_F}\right) e^{-i(\mu+1/2)\varphi} J_{\mu+1/2}(k_F \rho \cos \theta) \quad (6)$$

$$v_\mu(\mathbf{r}) = \left(\frac{k_F \Delta_0}{2L_z v_F}\right) e^{-i(\mu-1/2)\varphi} J_{\mu-1/2}(k_F \rho \cos \theta) \quad (7)$$

where J_p is a cylindrical Bessel function and L_z is the length of the vortex line. We define the characteristic length scale of the vortex core via the behavior of the order parameter near $\rho = 0$:

$$\lim_{\rho \rightarrow 0} \Delta(\mathbf{r}) = e^{-i\varphi} \Delta_0 \frac{\rho}{\xi_1}. \quad (8)$$

As $T \rightarrow 0$, the sum in Eq. (5) is dominated by the lowest positive energy state, $\mu = 1/2$. Inserting Eqs. (6) and (7) for $\mu = 1/2$, and using $J_p(x) \approx \frac{1}{p!} \left(\frac{x}{2}\right)^p$ for $x \ll 1$, we obtain the behavior of the order parameter near $\rho = 0$ from Eq. (5) as

$$\Delta(\mathbf{r}) = \frac{\pi|a|}{m_a} \frac{k_F}{L_z v_F} \Delta_0 e^{-i\varphi} \rho \sum_{k_z} \cos \theta \tanh\left(\frac{E_{1/2, k_z}}{2k_B T}\right). \quad (9)$$

By comparing with Eq. (8) we find

$$\frac{1}{\xi_1} = \frac{\pi k_F |a|}{L_z} \sum_{k_z} \cos \theta \tanh\left(\frac{E_{1/2, k_z}}{2k_B T}\right), \quad (10)$$

where we have used $v_F = k_F / m_a$. Converting the sum over k_z to an integral over θ through the relation

$$\sum_{k_z} = \frac{L_z}{2\pi} \int dk_z = \frac{L_z k_F}{2\pi} \int_0^\pi d\theta \cos \theta, \quad (11)$$

we find

$$\frac{1}{\xi_1} = \frac{k_F^2 |a|}{2} \int_0^\pi \cos^2 \theta d\theta = \frac{\pi k_F^2 |a|}{4}, \quad (12)$$

since $\tanh(E_{1/2, k_z} / 2k_B T) \approx 1$ as $T \rightarrow 0$. The characteristic length scale at low temperatures $T \ll T_c$ is therefore given by

$$\xi_1 = \frac{4}{\pi k_F^2 |a|}, \quad (13)$$

and is seen to be a result of the dominating role played by the low-lying eigenstates of the BdG Hamiltonian at low temperatures. Numerical results obtained by Gygi and Schlüter [15,16] have supported this conclusion. Furthermore, De Blasio and Elgarøy [17,18], in a study of vortices in superfluid neutron matter at $T = 0$, considered various ways of defining the vortex core size. They all gave characteristic sizes generally smaller than ξ_0 , the differences being particularly pronounced for low values of Δ_0 , which is the case in the dilute gas limit. The most important feature, however, was that they had a much weaker dependence on $k_F |a|$ than the BCS coherence length, and that the overall trend of the numerical results was well approximated by the Kramer-Pesch estimate for the vortex core size, ξ_1 . The ratio of this quantity to ξ_0 , $\xi_1 / \xi_0 \approx e^{-\pi/2 k_F |a| / k_F |a|}$, is plotted in Fig. 1, and can be seen to be much smaller than one in the weak coupling regime.

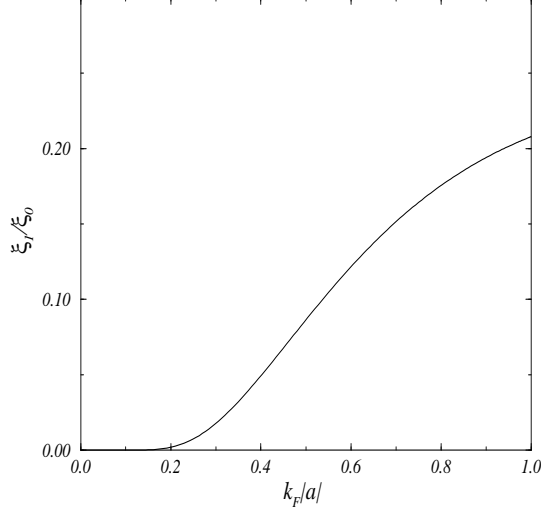


FIG. 1. The ratio ξ_1/ξ_0 of the Kramer-Pesch estimate for the vortex core size to the BCS coherence length.

To estimate the vortex energy per unit length at zero temperature, we follow the approach of [12], but with the important modification that we take the vortex core size to be given by ξ_1 instead of ξ_0 . The vortex is modeled as a cylindrical region of radius ξ_1 containing normal matter, surrounded by a velocity field with magnitude given by $v_s = \hbar/2m_a\rho$. The superfluid is confined within a cylinder of radius $R_c \gg \xi_1$, and is assumed to be uniform along the z -direction, which also defines the vortex axis. The energy per unit length \mathcal{E}_v of the vortex state with respect to the homogeneous superfluid can then be divided into two contributions. The first is the kinetic energy associated with the flow of the superfluid outside the vortex core, given by

$$\begin{aligned} \mathcal{E}_{\text{kin}} &= \int_{\xi_1}^{R_c} m_a n_\sigma \left(\frac{\hbar}{2m_a\rho} \right)^2 2\pi\rho d\rho \\ &= \frac{\pi\hbar^2 n_\sigma}{2m_a} \ln \left(\frac{R_c}{\xi_1} \right), \end{aligned} \quad (14)$$

and the second contribution, coming from the loss of condensation energy in the vortex core, can be estimated by multiplying the BCS result for the condensation energy per volume, $\epsilon_{\text{cond}} = 3\Delta_0^2 n_\sigma / 4\epsilon_F$, with the area of the vortex core $\pi\xi_1^2$. This gives

$$\mathcal{E}_{\text{cond}} = \frac{\pi\hbar^2 n_\sigma}{2m_a} \frac{3}{\pi^2} \left(\frac{\xi_1}{\xi_0} \right)^2. \quad (15)$$

The total energy can therefore be written in the usual form

$$\mathcal{E}_v = \frac{\pi\hbar^2 n_\sigma}{2m_a} \ln \left(D \frac{R_c}{\xi_0} \right), \quad (16)$$

however, in contrast with Eq. (2), D is now a function of $k_F|a|$ through the ratio ξ_1/ξ_0 . Since $\xi_1 \ll \xi_0$ in the weak coupling regime, we have

$$D = \frac{\xi_0}{\xi_1} \left[1 + \frac{3}{\pi^2} \left(\frac{\xi_1}{\xi_0} \right)^2 \right], \quad (17)$$

and the vortex energy is to a good approximation given by $\ln(R_c/\xi_1)$, which has also been found to be the case in neutron matter [18]. Note that D is essentially the inverse of the quantity plotted in Fig. 1, and so it increases rapidly with decreasing $k_F|a|$. At the limit of the region where weak coupling can reasonably be expected to hold, $k_F|a| \sim 0.4$, $D \approx 20.0$, which is an order of magnitude larger than in the estimate in Eq. (2).

We can now estimate the critical rotation frequency for a dilute gas of atoms trapped in a cylindrically symmetric harmonic oscillator potential

$$V(\mathbf{r}) = \frac{1}{2} m_a \omega_z^2 [z^2 + \lambda_T^2 (x^2 + y^2)], \quad (18)$$

where λ_T describes the anisotropy of the trap. We will use the Thomas-Fermi result for the density profile of this gas, given by

$$n_\sigma(\rho, z) = n_{\sigma,0} \left(1 - \frac{\lambda_T^2 \rho^2 + z^2}{R_z^2} \right)^{3/2}, \quad (19)$$

where $n_{\sigma,0} = n_\sigma(0,0)$ is the central density of the cloud, $R_z = (48N_\sigma\lambda_T^2)^{1/6}l_{\text{osc}}$ is the extent of the cloud in the z -direction, N_σ is the number of atoms in the hyperfine state σ , and $l_{\text{osc}} = \sqrt{\hbar/m_a\omega_z}$ [23]. Following Lundh *et al.* [13], we divide the cloud into vertical slices of height dz , and use (16) for a cylinder of radius ρ_1 such that $\xi_1 \ll \rho_1 \ll R_\perp = R_z/\lambda_T$, where the gas can be considered uniform. The energy per unit length of the slice at z can then be written as

$$\begin{aligned} \mathcal{E}_v &= \frac{\pi\hbar^2 n_\sigma(0,z)}{2m_a} \ln \left(\frac{\rho_1}{\xi_1(z)} \right) \\ &+ \int_{\rho_1}^{R_\perp(z)} m_a n_\sigma(\rho, z) \left(\frac{\hbar}{2m_a\rho} \right)^2 2\pi\rho d\rho, \end{aligned} \quad (20)$$

with $R_\perp(z) = (1 - z^2/R_z^2)^{1/2}R_z/\lambda_T$ being the extent of the cloud in the ρ -direction for a given z , and $n_\sigma(0,z)$ is the density along the z -axis. Inserting Eq. (19) and integrating, one finds

$$\begin{aligned} \mathcal{E}_v(z) &= \frac{\pi\hbar^2 n_{\sigma,0}}{2m_a} \left(1 - \frac{z^2}{R_z^2} \right)^{3/2} \\ &\times \ln \left[\frac{2}{e^{4/3}} \frac{R_z}{\lambda_T \xi_1(0)} \left(1 - \frac{z^2}{R_z^2} \right)^{3/2} \right], \end{aligned} \quad (21)$$

where $\xi_1(0)$ is the value of ξ_1 at the center of the cloud. Integrating over z , we obtain the total energy of the vortex as

$$E_v = \frac{\pi \hbar^2 n_{\sigma,0}}{2m_a} \frac{3\pi}{8} R_z \ln \left(0.379 \frac{R_z}{\lambda_T \xi_1(0)} \right). \quad (22)$$

The critical rotation frequency ω_{c1} for creating a vortex in the trap is given by $\omega_{c1} = E_v/L_v$ where $L_v = N_\sigma \hbar$ is the angular momentum of the vortex state. Using $N_\sigma = \pi^2 R_z^3 n_{\sigma,0}/8\lambda_T^2$, we find

$$\omega_{c1} = \frac{3}{2} \omega_\perp \frac{l_\perp^2}{R_\perp^2} \ln \left(0.379 \frac{R_\perp}{\xi_1(0)} \right), \quad (23)$$

where $l_\perp = l_{\text{osc}}/\lambda_T$. To compare with the result of Ref. [12], we take $\lambda_T = 1$ so that $\omega_z = \omega_\perp \equiv \omega$, and set $k_{F,0}|a| = 0.4$, $\epsilon_F = 200\hbar\omega$, corresponding to an isotropic trap with $N_\sigma \sim 1.3 \cdot 10^6$. With these values, we obtain $\omega_{c1} \approx 0.014\omega$, four times larger than in [12]. In Fig. (2) the ratio between the critical frequency found in this paper and the one found in [12] is plotted as a function of $k_F|a|$, with the other parameters kept at the same values as above. Note that ω_{c1} turns negative for small values of $k_F|a|$, reflecting that the condition $\xi_1 \ll R_\perp$ is violated. For the parameters used in this example, this occurs at $k_F|a| \approx 0.01$ for ω_{c1} given by Eq. (23), and for $k_F|a| \approx 0.33$ for the corresponding result in Ref. [12].

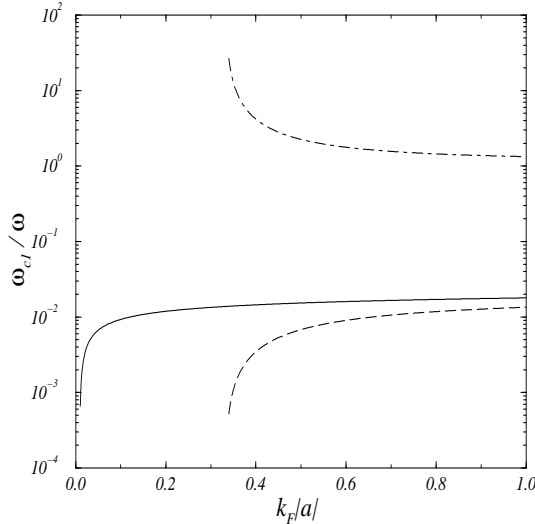


FIG. 2. The critical frequency, Eq. (23) (solid line), the critical frequency found in Ref. [10] (dashed line), and their ratio (dot-dashed line), for an isotropic trap with the parameters given in the text.

To conclude, we have estimated the energy of a vortex in a uniform, dilute gas of atoms at $T = 0$. The quasi-particle states in the core of the vortex introduce a new characteristic length scale ξ_1 which has a weaker density dependence than the BCS coherence length ξ_0 and is much smaller than ξ_0 in the dilute gas regime $k_F|a| \ll 1$.

Taking this fact into account leads us to predict a considerably higher vortex energy than what was found in earlier calculations. For the case of a trapped gas of fermionic atoms, this means that the critical frequency for the formation of a vortex state can be 5 – 10 times higher than previously estimated.

The author is grateful to F. V. de Blasio, G. M. Bruun, R. Kaldare, and C. J. Pethick for illuminating and valuable discussions. Support through a post-doctoral fellowship from The Research Council of Norway (NFR) is acknowledged.

-
- [1] H. Heiselberg and M. Hjorth-Jensen, Phys. Rep. **328**, 237 (2000).
 - [2] M. H. Anderson *et al.*, Science **269**, 198 (1995).
 - [3] H. T. C. Stoof, M. Houbiers, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **76**, 10 (1996).
 - [4] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999).
 - [5] B. DeMarco, S. B. Papp, and D. S. Jin, Phys. Rev. Lett. **86**, 5409 (2001).
 - [6] A. G. Truscott *et al.*, Science **291**, 2570 (2001).
 - [7] F. Schreck *et al.*, Phys. Rev. A **64**, 011402 (2001).
 - [8] K. M. O'Hara *et al.*, Phys. Rev. Lett. **85**, 2092 (2000).
 - [9] F. Weig and W. Zwerger, Europhys. Lett. **49**, 282 (2000); W. Zhang, C. A. Sackett, and R. G. Hulet, Phys. Rev. A **60**, 504(1999); J. Ruostekoski, Phys. Rev. A **60**, R1775 (1999); P. Törmä and P. Zoller, Phys. Rev. Lett. **85**, 487 (2000).
 - [10] A. L. Fetter and A. A. Svidzinsky, J. Phys.: Condens. Matter **13**, R135 (2001).
 - [11] M. Rodriguez, G.-S. Paraoanu, and P. Törmä, Phys. Rev. Lett. **87**, 100402 (2001).
 - [12] G. M. Bruun and L. Viverit, Phys. Rev. A **64**, 063606 (2001).
 - [13] E. Lundh, C. J. Pethick, and H. Smith, Phys. Rev. A **55**, 2126 (1997).
 - [14] L. Kramer and W. Pesch, Z. Physik **269**, 59 (1974).
 - [15] F. Gygi and M. Schlüter, Phys. Rev. B **41**, 822 (1990).
 - [16] F. Gygi and M. Schlüter, Phys. Rev. B **43**, 7609 (1991).
 - [17] F. V. de Blasio and Ø. Elgarøy, Phys. Rev. Lett. **82**, 1815 (1999).
 - [18] Ø. Elgarøy and F. V. de Blasio, Astron. & Astrophys. **370**, 939 (2001).
 - [19] L. P. Gorkov and T. K. Melik-Barkhudarov, Sov. Phys. JETP **13**, 1018 (1961).
 - [20] H. Heiselberg, C. J. Pethick, H. Smith, and L. Viverit, Phys. Rev. Lett. **85**, 2418 (2000).
 - [21] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, Reading, MA, 1989).
 - [22] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Lett. **9**, 307 (1964).
 - [23] D. A. Butts and D. S. Rokhsar, Phys. Rev. A **55**, 4346 (1997).